Physics 566: Quantum Optics

Problem Set 8: Solutions

Problem 1: Nonclassical light via the Kerr effect

The Optical Kerr effect involves an intensity dependent index of refraction

$$\Delta n = n_2 I \qquad n_2 \propto \chi^{(3)}$$

The induced polarization $P \propto \Delta n E$

$\Rightarrow$ Interaction energy $H \propto PE^* = \Delta n I^2$

Quantizing $\Rightarrow$ Normal order $H = \Delta n :I^2:$
(in appropriate units)

We take the quantum Hamiltonian:

$$\hat{H} = \frac{\hbar \chi^{(3)}}{2} \hat{a}^{\dagger 2} \hat{a}^2$$

(a) For a strong input we can linearize

$\Rightarrow$ Take $\hat{a} = \alpha + \hat{b}$ and keep terms only to quadratic order in $\hat{b}$

$$\Rightarrow \hat{H} = \frac{\hbar \chi^{(3)}}{2} (\alpha^* + \hat{b}^\dagger)^2 (\alpha + \hat{b})^2 \cong \frac{\hbar \chi^{(3)}}{2} (\alpha^{*2} + 2\hat{b}^\dagger \alpha^* + \hat{b}^{\dagger 2}) \times (\alpha^2 + 2\hat{b}\alpha + \hat{b}^2)$$

$$\cong \frac{\hbar \chi^{(3)}}{2} \Big[|\alpha|^4 + 2|\alpha|^2 (\alpha \hat{b}^\dagger + \alpha^* \hat{b}) + 4|\alpha|^2 \hat{b}^\dagger \hat{b} + \left(\alpha^2 \hat{b}^{\dagger 2} + (\alpha^*)^2 \hat{b}^2 \right) \Big]$$

The Hamiltonian exhibits ~~three~~ four terms

- $\frac{\hbar\chi^{(3)}}{2}|\alpha|^4$: Constant, classical Kerr effect

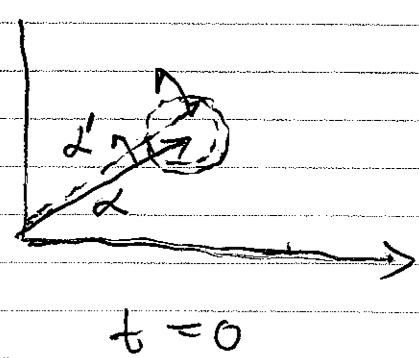
- $\hbar\,\chi^{(3)}|\alpha|^2(\alpha\hat{b}^\dagger + \alpha^*\hat{b})$: Phase-space displacement (parametric

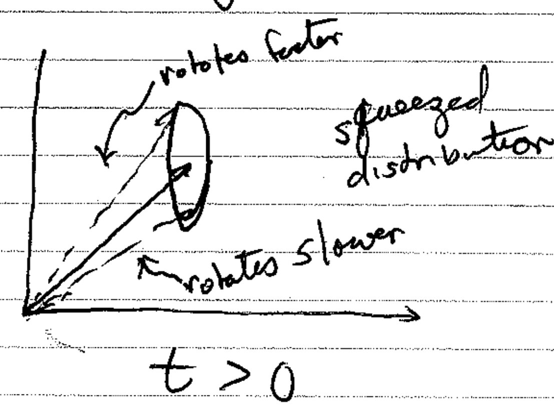
- $2\hbar\,\chi^{(3)}|\alpha|^2\,\hat{b}^\dagger\hat{b}$: Cross-phase modulation between pump and fluctuations $\Rightarrow$ rotation in phase space

- $\frac{\hbar\chi^{(3)}}{2}|\alpha|^2(\hat{b}^{\dagger 2}e^{2i\phi} + \hat{b}^2 e^{-2i\phi})$: Squeezing Hamiltonian

Thus, to this order we see that the Kerr effect will lead to squeezing. Because of the cross-phase modulation the axis of squeezing will continually rotate.

A simple classical picture explains this. Due to the Kerr effect, a phasor will rotate <u>faster</u> the longer the vector (bigger intensity)





(b) The "shearing" of the uncertainty bubble eventually distorts it beyond elliptical. At this point, the linear approximation breaks down.

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}t}\,|\psi(0)\rangle$$

$$|\psi(0)\rangle = |\alpha\rangle = \sum_n c_n |n\rangle \qquad c_n = \frac{(\alpha)^n}{\sqrt{n!}} e^{-|\alpha|^2/2}$$

$$\hat{H} = \frac{\hbar\chi^{(3)}}{2}\hat{a}^{\dagger 2}\hat{a}^2 = \frac{\hbar\chi^{(3)}}{2}\left[(\hat{a}^\dagger\hat{a})^2 - \hat{a}^\dagger a\right]$$

$$= \frac{\hbar\chi^{(3)}}{2}\left(\hat{n}^2 - \hat{n}\right)$$

$$\therefore\ |\psi(t)\rangle = \sum_n e^{-i\frac{\chi^{(3)}t}{2}(n^2-n)}\, c_n|n\rangle$$

when $\chi^{(3)}t = \pi$

$$|\psi(t)\rangle = \sum_n e^{-i\frac{\pi}{2}(n^2-n)}\, c_n|n\rangle$$

$$= e^{-\frac{|\alpha|^2}{2}}\sum_n (-i)^{n^2}\frac{(+i\alpha)^n}{\sqrt{n!}}|n\rangle$$

Now $(-i)^{n^2} = \begin{cases} 1 & n\ \text{even} \\ -i & n\ \text{odd}\end{cases} = \frac{1}{\sqrt{2}}\left(e^{-i\pi/4} + (-1)^n e^{i\pi/4}\right)$

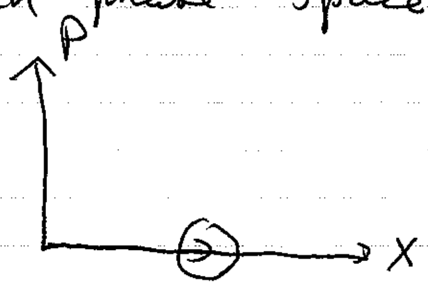
$$\Rightarrow |\psi(t)\rangle = \frac{e^{-i\pi/4}}{\sqrt{2}}e^{-|\alpha|^2/2}\sum_n \frac{(i\alpha)^n}{\sqrt{n!}}|n\rangle$$

$$+\frac{e^{i\pi/4}}{\sqrt{2}}e^{-|\alpha|^2/2}\sum_n \frac{(-i\alpha)^n}{\sqrt{n!}}|n\rangle$$

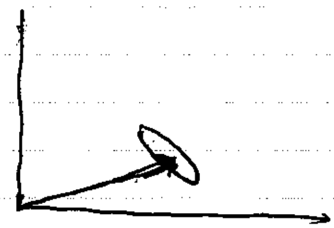
$$\Rightarrow \boxed{|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\pi/4}\,|i\alpha\rangle + e^{i\pi/4}\,|-i\alpha\rangle \right)}$$

This is a so-called "Schrödinger cat state".

In phase space:

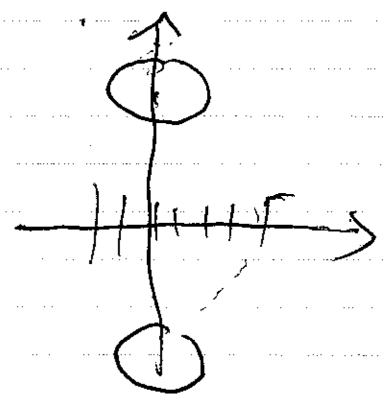


$t=0$ (take α real)



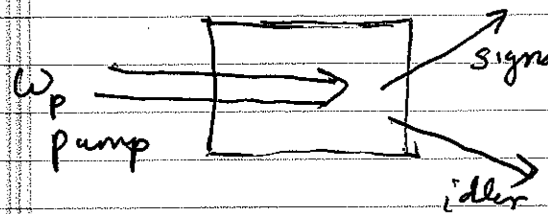
t small (squeezed)

t larger
(distorted ellipse
(negative Wigner function)



Schrodinger cat!

Problem 2: Two mode squeezed states



$$\omega_+ = \frac{\omega_p}{2} + \Delta\omega$$

$$\omega_- = \frac{\omega_p}{2} - \Delta\omega$$

"Twin beams"
Quantum correlated

$$\hat{H} = i\hbar G\left(\hat{a}_+^\dagger \hat{a}_-^\dagger e^{-i\phi} - \hat{a}_+ \hat{a}_- e^{i\phi}\right)$$

Time evolution operator: $\hat{U}(t) = e^{-i\hat{H}t}$

$$\hat{U}(t) = e^{\xi \hat{a}_+ \hat{a}_- - \xi^* \hat{a}_+^\dagger \hat{a}_-^\dagger} = \hat{S}(\xi)$$

$$\xi = r e^{i\phi} \qquad r = Gt$$

(a) Generalized Bogoliubov transformation

$$\hat{S}^\dagger(\xi)\,\hat{a}_\pm\,\hat{S}(\xi) = \hat{a}_\pm + \frac{it}{\hbar}\left[\hat{H}, \hat{a}_\pm\right] + \frac{1}{2}\left(\frac{it}{\hbar}\right)^2 \left[\hat{H},\left[\hat{H},\hat{a}_\pm\right]\right] + \dots$$

(Baker – Campbell – Hausdorff)

Aside: $\left[\hat{H}, \hat{a}_\pm\right] = -i\hbar G \hat{a}_\mp^\dagger e^{-i\phi}$

$$\left[\hat{H},\left[\hat{H},\hat{a}_\pm\right]\right] = -i\hbar G e^{-i\phi}\left[\hat{H}, a_\mp^\dagger\right]$$

$$= \left(-i\hbar G e^{-i\phi}\right)\left(i\hbar G e^{i\phi}\right)\hat{a}_\pm$$

$$= -(\hbar G)^2\,\hat{a}_\pm$$

etc.

$$\therefore\ \hat{S}^{\dagger}(\xi)\,\hat{a}_{\pm}\,\hat{S}(\xi) = \left(\sum_{n\ \text{even}} \frac{r^n}{n!}\right)\hat{a}_{\pm} - e^{-i\phi}\left(\sum_{n\ \text{odd}} \frac{r^n}{n!}\right)\hat{a}^{\dagger}_{\mp}$$

$$\Rightarrow \boxed{\hat{S}^{\dagger}(\xi)\,\hat{a}_{\pm}\,\hat{S}(\xi) = \cosh(r)\,\hat{a}_{\pm} - e^{-i\phi}\sinh(r)\,\hat{a}^{\dagger}_{\mp}}$$

(b) Consider quadrature operators:

$$\hat{X}_{\pm} \equiv \frac{\hat{a}_{\pm}e^{+i\theta} + \hat{a}^{\dagger}_{\pm}e^{-i\theta}}{2} \quad \text{for each mode}$$

$$\hat{S}^{\dagger}\hat{X}_{\pm}(\theta)\hat{S} = \frac{(c\hat{a}_{\pm} - e^{-i\phi}s\hat{a}^{\dagger}_{\mp})e^{+i\theta} + (c\hat{a}^{\dagger}_{\pm} - e^{i\phi}s\hat{a}_{\mp})e^{-i\theta}}{2}$$

$$= c\left(\frac{\hat{a}_{\pm}e^{+i\theta} + \hat{a}^{\dagger}_{\pm}e^{-i\theta}}{2}\right) - s\left(\frac{\hat{a}_{\mp}e^{-i(\theta-\phi)} \pm \hat{a}^{\dagger}_{\mp}e^{+i(\theta-\phi)}}{2}\right)$$

$$= c\,\hat{X}_{\pm}(\theta) - s\,\hat{X}_{\mp}(\theta-\phi) \qquad \begin{pmatrix} c \equiv \cosh r \\ s \equiv \sinh r \end{pmatrix}$$

Thus if we look at the fluctuations in these quadratures

$$\Delta\hat{X}_{\pm}^{2} = \langle \hat{X}_{\pm}^{2} \rangle - \langle \hat{X}_{\pm} \rangle^{2}$$

(where the expectation value is taken in squeezed vacuum)

we will find no reduction below shot noise

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$$|0_\xi\rangle \equiv \hat{S}(\xi)\,|0\rangle_+ \otimes |0\rangle_-$$

$$\Rightarrow \langle 0_\xi|\,\hat{X}_\pm\,|0_\xi\rangle = 0$$

$$\langle 0_\xi|\,\hat{X}_\pm^2\,|0_\xi\rangle = \langle 0_+| \otimes \langle 0_-|\,\left(c\,\hat{X}_\pm(\theta) - s\,\hat{X}_\mp(\theta-\phi)\right)^2\,|0\rangle_+ \otimes |0\rangle_-$$

$$= c^2 \underbrace{\langle 0_+ 0_-|\,\hat{X}_\pm^2(\theta)\,|0_+ 0_-\rangle}_{\frac{1}{4}} + s^2 \underbrace{\langle 0_+ 0_-|\,\hat{X}_\mp(\theta-\phi)\,|0_+ 0_-\rangle}_{\frac{1}{4}}$$

$$= \frac{1}{4}\left(\cosh^2(r) + \sinh^2(r)\right) > \frac{1}{4}$$

$\Rightarrow$ No squeezing in single beams
(in fact extra noise in each beam)

Now consider the <u>noise difference</u>

$$\hat{Y}(\theta,\theta') \equiv \hat{X}_+(\theta) - \hat{X}_-(\theta')$$

$$\Rightarrow \hat{S}^\dagger \hat{Y}(\theta,\theta')\hat{S} = \left(c\,\hat{X}_+(\theta) - s\,\hat{X}_-(\theta-\phi)\right) - \left(c\,\hat{X}_-(\theta') - s\,\hat{X}_+(\theta'-\phi)\right)$$

$$= \left[c\,\hat{X}_+(\theta) + s\,\hat{X}_+(\theta'-\phi)\right]$$

$$- \left[c\,\hat{X}_-(\theta') + s\,\hat{X}_-(\theta-\phi)\right]$$

$$\left\langle \Delta \hat{Y}^2(\theta,\theta')\right\rangle = \left\langle \left(c\hat{X}_+(\theta) + s\hat{X}_+(\theta'-\phi)\right)^2\right\rangle$$
$$+\left\langle \left(c\hat{X}_-(\theta) + s\hat{X}_-(\theta-\phi)\right)^2\right\rangle$$

Aside:

$$\left\langle \left(c\hat{X}_+(\theta) + s\hat{X}_+(\theta'-\phi)\right)^2\right\rangle$$
$$= c^2\langle \hat{X}_+^2(\theta)\rangle + s^2\langle \hat{X}_+^2(\theta'-\phi)\rangle$$
$$+ cs\left(\langle \hat{X}_+(\theta)\hat{X}_+(\theta'-\phi)\rangle + \langle \hat{X}_+(\theta'-\phi)\hat{X}_+(\theta)\rangle\right)$$

$$\langle \hat{X}_+^2(\theta)\rangle = \langle \hat{X}_+^2(\theta'-\phi)\rangle = \frac{1}{4}$$

$$\langle \hat{X}_+(\theta)\hat{X}_+(\theta'-\phi)\rangle = \left\langle \left(\frac{\hat{a}_+e^{i\theta} + \hat{a}_+^\dagger e^{-i\theta}}{2}\right)\left(\frac{\hat{a}_+e^{i(\theta'-\phi)} + \hat{a}_+^\dagger e^{-i(\theta'-\phi)}}{2}\right)\right\rangle$$

$$= \frac{1}{4}\langle \hat{a}_+\hat{a}_+^\dagger\rangle e^{i(\theta-\theta'-\phi)}$$

$$\therefore = \frac{c^2+s^2+2cs\cos(\theta-\theta'-\phi)}{4}$$

$$= \frac{1}{4}\left[e^{-2r}\sin^2\left(\frac{\theta-\theta'-\phi}{2}\right) + e^{2r}\cos^2\left(\frac{\theta-\theta'-\phi}{2}\right)\right]$$

$$\therefore \boxed{\left\langle \Delta \hat{Y}^2(\theta,\theta')\right\rangle = \frac{1}{2}\left[e^{-2r}\sin^2\left(\frac{\theta-\theta'-\phi}{2}\right) + e^{2r}\cos^2\left(\frac{\theta-\theta'-\phi}{2}\right)\right]}$$

Interpretation:

When $\theta - \theta' - \phi = \pi$, the modes are maximally correlated $\Rightarrow$ Squeezing

(c) Number correlations in twin beams:

$$\hat{S}_{\pm}(r)\,|0_+\rangle|0_-\rangle = e^{r(\hat{a}_+^\dagger \hat{a}_-^\dagger - \hat{a}_+\hat{a}_+)}\,|0_+\rangle\otimes|0_-\rangle$$

take ξ real

To apply this operator, we need the disentangling theorem (see Walls + Milburn, eq. (5.63))

$$\hat{S}_{\pm}(r) = e^{\Gamma \hat{a}_+^\dagger \hat{a}_-^\dagger}\, e^{-g(\hat{a}_+^\dagger \hat{a}_+ + \hat{a}_-^\dagger \hat{a}_- + 1)}\, e^{-\Gamma \hat{a}_+ \hat{a}_-}$$

where $\Gamma = \tanh(r)$ $\quad g = \ln(\cosh r)$

Applying this to the vacuum

$$\hat{S}_{\pm}(r)\,|0_+\rangle\otimes|0_-\rangle = e^{\Gamma \hat{a}_+^\dagger \hat{a}_-^\dagger}\, e^{-g}\,|0_+\rangle\otimes|0_-\rangle$$

$$= \frac{1}{e^g}\sum_n \Gamma^n \frac{(\hat{a}_+^\dagger)^n}{\sqrt{n!}}\frac{(\hat{a}_-^\dagger)^n}{\sqrt{n!}}\,|0_+\rangle\otimes|0_-\rangle$$

$$\Rightarrow \boxed{\hat{S}_{\pm}(r)\,|0_+\rangle\otimes|0_-\rangle = \frac{1}{\cosh(r)}\sum_n [\tanh(r)]^n\,|n_+\rangle|n_-\rangle}$$

Note for r small, we get the correlated photon pair (plus vacuum) d)

To obtain the margin density operator we trace over one of the degrees of freedom, e.g.,

$$\hat{\rho}_+ = \mathrm{Tr}_-\left(|\psi_{+-}\rangle\langle\psi_{+-}| \right)$$
$$= \frac{1}{\cosh^2(r)} \sum_n{}' \left[\tanh^2(r)\right]^n |n\rangle_+\langle n|$$

this has the form of a thermal state

$$\hat{\rho}_{thermal} = \sum_{n=0}^{\infty}{}' P_n |n\rangle\langle n|$$

where $P_n = \dfrac{\left(e^{-\beta\hbar\omega}\right)^n}{Z} = \dfrac{\bar{n}^n}{(1+\bar{n})^{n+1}}$ is the Bose-Einstein distribution

setting $\left(\dfrac{\bar{n}}{1+\bar{n}}\right)^n \dfrac{1}{1+\bar{n}} = \dfrac{\left(\tanh^2(r)\right)^n}{\cosh^2(r)}$

using $\sinh^2(r) - \cosh^2(r) = 1$

$\tanh^2(r) - 1 = \dfrac{1}{\cosh^2(r)}$

$$\Longrightarrow \boxed{\bar{n} = \sinh^2(r)}$$

(c) Wigner function of the two mode squeezed state

For two modes

$$W(\alpha_+, \alpha_-) = \frac{1}{\pi^4} \int d^2\beta_+ \int d^2\beta_- \, \chi(\beta_+, \beta_-) \, e^{\alpha_+ \beta_+^* - \alpha_+^* \beta_+} \times e^{\alpha_- \beta_-^* - \alpha_-^* \beta_-}$$

where the characteristic function is

$$\chi(\beta_+, \beta_-) = \mathrm{Tr}(\hat{\rho}\, \hat{D}_+(\beta_+)\, \hat{D}_-(\beta_-))$$

$$= \langle 0_r | \hat{D}_+(\beta_+) \hat{D}_-(\beta_-) | 0_r \rangle$$

$$|0_r\rangle = \exp\{ r(\hat{a}_+ \hat{a}_- - \hat{a}_+^\dagger \hat{a}_-^\dagger) \} |0_+\rangle \otimes |0_-\rangle$$

$$\Rightarrow \chi(\beta_+, \beta_-) = \langle 0_r | e^{\beta_+ \hat{a}_+^\dagger - \beta_+^* \hat{a}_+} e^{\beta_- \hat{a}_-^\dagger - \beta_-^* \hat{a}_-} | 0_r \rangle$$

$$= \langle 0_r | e^{(c\beta_+ - s\beta_-^*)\hat{a}_+^\dagger - (c\beta_+^* - s\beta_-)\hat{a}_+} \, e^{(c\beta_- - s\beta_+^*)\hat{a}_-^\dagger - (c\beta_-^* - s\beta_+)\hat{a}_-} | 0_r \rangle$$

$$= \langle 0_+ | \hat{D}_+(c\beta_+ - s\beta_-^*) | 0_+ \rangle \langle 0_- | \hat{D}_-(c\beta_- - s\beta_+^*) | 0_- \rangle$$

$$= e^{-\frac{1}{2}|c\beta_+ - s\beta_-^*|^2} \, e^{-\frac{1}{2}|c\beta_- - s\beta_+^*|^2}$$

Fourier transforming, using our Gaussian integral from PS#4

$$W(\alpha_+, \alpha_-) = \frac{4}{\pi^2} \exp\left[-2|\alpha_+ \cosh r - \alpha_-^* \sinh r|^2 - 2|\alpha_- \cosh r - \alpha_+^* \sinh r|^2\right]$$

In terms of the quadrature $x_\pm + i p_\pm \equiv \alpha_\pm$

$$W(x_\pm, p_\pm) = \frac{4}{\pi^2} \exp\Big[-2\,(x_+ c - x_- s)^2 - 2\,(p_+ c + p_- s)^2 - 2\,(x_- c - x_+ s)^2 - 2\,(p_- c + p_+ s)^2\Big]$$

$$= \frac{4}{\pi^2} \exp\Big[-2(x_+^2 + x_-^2 + p_+^2 + p_-^2)\,(c^2 + s^2) + 8(x_+ x_- - p_+ p_-)\, c s\Big]$$

with $c^2 + s^2 = \frac{e^{2r} + e^{-2r}}{2}$ $\qquad cs = \frac{e^{2r} - e^{-2r}}{4}$

$$\Rightarrow W(x_\pm, p_\pm) = \frac{4}{\pi^2} \exp\Big[-e^{+2r}\left(x_+^2 + x_-^2 - 2x_+ x_- + p_+^2 + p_-^2 + 2p_+ p_-\right) - e^{-2r}\left(x_+^2 + x_-^2 + 2x_+ x_- + p_+^2 + p_-^2 - 2p_+ p_-\right)\Big]$$

$$\Rightarrow \boxed{W(x_\pm, p_\pm) = \frac{4}{\pi^2} \exp\Big[-e^{-2r}\{(x_+ - x_-)^2 + (p_+ + p_-)^2\} - e^{+2r}\{(x_+ + x_-)^2 + (p_+ - p_-)^2\}\Big]}$$

In the limit $r \to \infty$ the squeezed direction goes to zero and the stretched direction to ∞, yielding a delta function

$$\boxed{W(x_\pm, p_\pm) \longrightarrow C\ \delta(x_+ + x_-)\ \delta(p_+ - p_-)}$$

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Problem 3: Gaussian States in Quantum Optics

We consider the state associated with n modes of the quantized electromagnetic field. They are described by n-pairs of quadratures ordered as

$$\vec{Z} = (X_1, P_1, X_2, P_2, \dots X_n, P_n)$$

the operators satisfy the canonical commutation relations

$$[\hat{Z}_i, \hat{Z}_j] = \frac{i}{2} \Sigma_{ij}$$

where

$$\Sigma = \begin{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} & & \\ & \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} & \\ & & \ddots \end{bmatrix}$$

(a) For a given mode, we defined the phase-space displacement operator

$$\hat{D}(\alpha) = e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}} = e^{-2i(X\hat{P} - P\hat{X})}$$

where $\alpha = X + iP$ $\quad \hat{a} = \hat{X} + i\hat{P}$

Note: $$[X \ \ P]\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \hat{X} \\ \hat{P} \end{bmatrix} = X\hat{P} - P\hat{X} \equiv (\vec{Z} | Z)$$

$$\Rightarrow \hat{D}(\alpha) = e^{-2i(\vec{Z}|Z)}$$

More generally, for multi mode $\hat{D}(\vec{Z}) = \prod_k \hat{D}_k(\alpha_k)$ ✓

The characteristic function for a general Gaussian state has the form:

$$\chi(\vec{z}) \equiv \mathrm{Tr}(\hat{\rho}\,\hat{D}(\vec{z})) = e^{-2(\vec{z}|C|\vec{z}) + 2i(\vec{d}|\vec{z})}$$

where $(\vec{z}|C|\vec{z}) = z_i \Sigma_{ij} C_{jk} \Sigma_{kl} z_l$ (sum over repeated indecies)

$(\vec{d}|\vec{z}) = d_i \Sigma_{ij} z_j$

For clarity, consider one mode first and let $\vec{d} = (X_0, P_0)$ $\quad C = \begin{bmatrix} C_{xx} & C_{xp} \\ C_{px} & C_{pp} \end{bmatrix}$

$$\Rightarrow \chi(X,P) = \exp\left\{-2\left(P^2 C_{xx} + X^2 C_{xp} - XP(C_{xp} + C_{px})\right)\right\} e^{2i(X_0 P - P_0 X)}$$

$$\Rightarrow \frac{\partial}{\partial X}\chi(X,P)) = \left(-4P\,C_{xx} + 2X(C_{xp} + C_{px}) - 2i P_0\right)\chi(X,P)$$

$$\frac{\partial}{\partial P}\chi(X,P) = \left(-4X\,C_{pp} + 2P(C_{xp} + C_{px}) + 2i X_0\right)\chi(X,P)$$

$$\Rightarrow \frac{\partial}{\partial X}\chi(\vec{z})\Big|_{\vec{z}=0} = -2iP_0 \qquad \frac{\partial}{\partial P}\chi(\vec{z})\Big|_{\vec{z}=0} = 2iX_0$$

Now, $\frac{\partial}{\partial x}\chi(\vec{z})\Big|_{\vec{z}=0} = \frac{\partial}{\partial x}\mathrm{Tr}\left(\hat{\rho}\, e^{-2i(x\hat{p}-p\hat{x})}\right)\Big|_{\vec{z}=0}$

$$= -2i\,\mathrm{Tr}(\hat{\rho}\,\hat{p}) = -2i\langle\hat{p}\rangle$$

$$\frac{\partial}{\partial p}\chi(\vec{z})\Big|_{\vec{z}=0} = \frac{\partial}{\partial p}\mathrm{Tr}\left(\hat{\rho}\, e^{-2i(x\hat{p}-p\hat{x})}\right)\Big|_{\vec{z}=0}$$

$$= 2i\,\mathrm{Tr}(\hat{\rho}\,\hat{x}) = 2i\langle\hat{x}\rangle$$

$\Rightarrow \langle\hat{x}\rangle = x_0 \qquad \langle\hat{p}\rangle = p_0$

or $\boxed{\langle\hat{\vec{z}}\rangle = \vec{d}}$ ✓

For the second derivatives we see

$$\frac{\partial^2}{\partial x^2}\chi(x,p)\Big|_{\vec{z}=0} = -4C_{pp} - 4p_0^2$$

$$\frac{\partial^2}{\partial p^2}\chi(x,p)\Big|_{\vec{z}=0} = -4C_{xx} - 4x_0^2$$

$$\frac{\partial^2}{\partial x\,\partial p}\chi(x,p)\Big|_{\vec{z}=0} = 2(C_{xp} + C_{px}) + 4x_0 p_0$$

And: $\frac{\partial^2}{\partial x^2}\chi(x,p)\Big|_{\vec{z}=0} = \frac{\partial^2}{\partial x^2}\mathrm{Tr}(\hat{\rho}\,D(\vec{z}))\Big|_{\vec{z}=0} = -4\langle\hat{p}^2\rangle$

$$\frac{\partial^2}{\partial x^2}\chi(x,p)\Big|_{\vec{z}=0} = \frac{\partial^2}{\partial p^2}\mathrm{Tr}(\hat{\rho}\,D(\vec{z}))\Big|_{\vec{z}=0} = -4\langle\hat{x}^2\rangle$$

$$\frac{\partial^2}{\partial x\,\partial p}\chi(x,p)\Big|_{\vec{z}=0} = \frac{\partial^2}{\partial x\,\partial p}\mathrm{Tr}(\hat{\rho}\,D(\vec{z}))\Big|_{\vec{z}=0} = 2\langle\hat{x}\hat{p} + \hat{p}\hat{x}\rangle$$

Thus we see:

$$C_{pp} = \langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2$$

$$C_{xx} = \langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2$$

$$C_{xp} = C_{px} = \left\langle \frac{\hat{x}\hat{p} + \hat{p}\hat{x}}{2} \right\rangle - \langle \hat{x} \rangle \langle \hat{p} \rangle$$

Or generally $\underset{\text{Covariance matrix}}{C_{ij}} = \frac{1}{2}\langle (\hat{z}_i - \langle \hat{z}_i \rangle)(\hat{z}_j - \langle \hat{z}_j \rangle) \rangle$

this generalizes for multi-modes if we just keep track of all the indeces.

(c) The Wigner function is the Fourier transform of the characteristic function.

$$W(\vec{z}) = \int \frac{dz'}{\pi^n} \chi(\vec{z}') e^{2i(\vec{z}'|\vec{z})}$$

Again, let us consider this for one mode

$$W(x,p) = \int \frac{dx\, dp}{\pi} \chi(x', p') e^{2i(xp' - x'p)}$$

The fourier transform of a Gaussian is a Gaussian. In 1D

$$f(x) = A e^{-\frac{(x-x_0)^2}{2\sigma^2}} \Rightarrow \tilde{f}(k) = A \frac{\sqrt{2\pi}}{\sigma} e^{-\frac{\sigma^2 k^2}{2}} e^{ikx_0}$$

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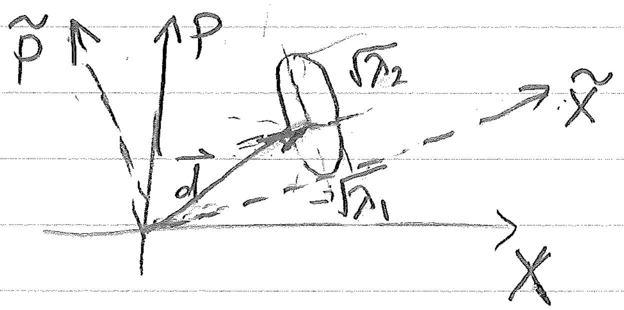
What we have is a multi-variable Gaussian. The key is that if we diagonalize the covariance matrix then the multivariate Gaussian factorizes, and we have a product of 1D Gaussians.

For example, for one mode, let λ_1 and λ_2 by the eigenvalues of the covariance matrix, and $\tilde{X}$, $\tilde{P}$ be the eigen-quadratures



λ_1 and λ_2 are the variances along $\tilde{X}$ and $\tilde{P}$

$\Rightarrow$ If the Characteristic function is

$$\chi(\tilde{X},\tilde{P}) = e^{-2(\lambda_1 \tilde{X}^2 + i d_{\tilde{x}} \tilde{X})} \, e^{-2(\lambda_2 \tilde{P}^2 - i d_{\tilde{p}} \tilde{P})}$$

The Wigner function is

$$W(\tilde{X},\tilde{P}) = \frac{2\pi}{\sqrt{\lambda_1 \lambda_2}} \, e^{-\frac{(\tilde{X} - d_{\tilde{x}})^2}{2\lambda_1}} \, e^{-\frac{(\tilde{P} - d_{\tilde{p}})^2}{2\lambda_2}}$$

Note in the eigenbasis, the covariance matrix

$$C = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \Rightarrow C^{-1} = \begin{bmatrix} \frac{1}{\lambda_1} & 0 \\ 0 & \frac{1}{\lambda_2} \end{bmatrix}$$

For an arbitray number of modes, this generalizes to

$$W(\tilde{Z}) = \frac{(2\pi)^n}{\sqrt{\lambda_1\lambda_2 \cdots \lambda_{1n}\lambda_{2n}}} \prod_{i=1}^{n} e^{-\frac{(\tilde{x}_i - d_{\tilde{x}_i})^2}{2\lambda_{1i}}} e^{-\frac{(\tilde{p} - d_{\tilde{p}_i})^2}{\lambda_{2i}}}$$

Note: $\prod_i \lambda_i = \det(C)$ = determinant of covariance matrix

If $\vec{d} = 0$ we can also compactly write the Wigner function in terms of the original quadratures

$$W(\vec{z}) = \frac{(2\pi)^n}{\sqrt{\det C}} e^{-\frac{1}{2}(\vec{z}|C^{-1}|\vec{z})}$$

inverse of covariance matrix

(d) We consider the unitary transformations that perform a symplectic map

$$\hat{U}^{\dagger} \hat{z}_i \hat{U} = S_{ij} \hat{z}_j \quad (\text{sum over } j)$$

We want to show how the covariance matrix of a state transforms.

Given a state $\hat{\rho}$, $\chi(\vec{z}) = \mathrm{Tr}(\hat{\rho}\hat{D}(\vec{z}))$

The characterist function of a transformed state $\hat{\rho}' = \hat{U}\hat{\rho}\hat{U}^{\dagger}$

is $\chi'(\vec{z}) = \mathrm{Tr}(\hat{U}\hat{\rho}\hat{U}^{\dagger}\hat{D}(\vec{z})) = \mathrm{Tr}(\hat{\rho}\,\hat{U}^{\dagger}\hat{D}(\vec{z})\hat{U})$

$$\mathrm{Tr}(\hat{\rho}\,\hat{U}\exp\{-2i\, z_i S_{ij} \hat{z}_j\}\hat{U}^{\dagger})$$

$$\Rightarrow \chi'(\vec{z}) = \mathrm{Tr}\left(\hat{\rho}\, \exp\{-2i\, z_i \Sigma_{ij}\, \hat{U} \hat{z}_j \hat{U}^\dagger\}\right)$$

$$= \mathrm{Tr}\left(\hat{\rho}\, \exp\{-2i\, z_i \Sigma_{ij} S^{-1}_{jk} \hat{z}_k\}\right)$$

Aside: $S^{-1} = \Sigma^{-1} S^T \Sigma$
(property of symplectic matrices)

Thus $\chi'(\vec{z}) = \mathrm{Tr}\left(\hat{\rho}\, \exp\{-2i\, z_i S^T_{ij} \Sigma_{jk} \hat{z}_k\}\right)$

$$= \mathrm{Tr}\left(\hat{\rho}\, \exp\{-2i\, (S\vec{z}|\hat{\vec{z}})\}\right)$$

$$\Rightarrow \boxed{\chi'(\vec{z}) = \chi(S\vec{z})}$$

The new characteristic function is obtained by transforming the $\vec{z} \Rightarrow S\vec{z}$

For Gaussian State: $\chi(\vec{z}) = e^{-2 z_i \Sigma_{ij} C_{jk} \Sigma_{kl} z_l}$
Upon a symplectic transformation

$$\chi'(\vec{z}) = \chi(S\vec{z}) = e^{-2 S_{il} z_l \Sigma_{ij} C_{jk} \Sigma_{kl} S_{lm} z_m}$$

$$= e^{-2 z_l S^T_{li} \Sigma_{ij} C_{jk} \Sigma_{kl} S_{lm} z_m}$$

Aside: For symplectic matrices $S^T \Sigma = \Sigma S$

$$\Rightarrow \chi'(\vec{z}) = e^{-2 z_i \Sigma_{ij} (S_{jk} C_{kl} S^T_{lm}) \Sigma_{mn} z_n}$$

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Thus, under a symplectic transformation, the covariance matrix of the Gaussian transforms as:

$$\boxed{C \Rightarrow S C S^T}$$

(e) Now we consider simple transformations

- Linear optics: $\hat{U} = e^{-i\theta_{ij}\hat{a}_i^\dagger \hat{a}_j}$ Hermitian matrix

$$\Rightarrow \hat{U} = \prod_\alpha e^{-i\theta_\alpha \hat{a}_\alpha^\dagger \hat{a}_\alpha}$$ new modes

where θ_α are eigenvalues of θ_{ij}

$$\Rightarrow \hat{U}^\dagger \hat{a}_\alpha \hat{U} = e^{-i\theta_\alpha}\hat{a}_\alpha$$

or $\begin{cases} \hat{X}_\alpha = \cos\theta_\alpha \hat{X}_\alpha + \sin\theta_\alpha \hat{P}_\alpha \\ \hat{P}_\alpha = \cos\theta_\alpha \hat{P}_\alpha - \sin\theta_\alpha \hat{X}_\alpha \end{cases}$

symplectic transformation !

- Squeezing $\hat{U} = \exp\left(\zeta_{ij}^* \hat{a}_i \hat{a}_j - \zeta_{ij}\hat{a}_i^\dagger \hat{a}_j^\dagger\right)$

$$\hat{U}^\dagger \hat{a}_i \hat{U} = \cosh(r_{ij})\hat{a}_i - e^{2i\phi_{ij}}\sinh(r_{ij})\hat{a}_j^\dagger$$

where $\zeta_{ij} = r_{ij} e^{i2\phi_{ij}}$

$\Rightarrow$ Squeezing and anti-squeezing along some quadratures

This is a symplectic transformation as it is linear in $\hat{a}$ and $\hat{a}^{\dagger}$ and preserves the commutation relation

(f) The symplectic transformations are the linear transformations on the quadratures

- For the case of linear optics, the matrix S_{ij} an orthogonal matrix;

$$S^T S = \mathbb{1},$$

Since it corresponds to general rotation in the multidimensal phase space

- For the case of squeezing

The general symplectic matrix can be decomposed in so-called "Euler form"

$$S = O_1 \left(\bigoplus_{j=1}^{n} \begin{bmatrix} e^{r_j} & 0 \\ 0 & e^{-r_j} \end{bmatrix} \right) O_2$$

The matricies O_1 and O_2 are orthogonal rotations of the type arising in linear optics. the matrix

$$\bigoplus_{j=1}^{n} \begin{bmatrix} e^{r_j} & 0 \\ 0 & e^{-r_j} \end{bmatrix}$$

performs squeezing along along X, P quadratures

Stated in another way:

$$S = \underbrace{O_3 \; O_2^T \left(\bigoplus_j \begin{bmatrix} e^{r_j} & 0 \\ 0 & e^{-r_j} \end{bmatrix} \right) O_2}_{\text{Similarity transformation}}$$

Here, I have written $O_1 = O_3 O_2^T$
Geometrically, this corresponds to rotating the principle axes of the ellipse to X-P, applying squeezing, and then a final rotation

(f) - For linear optics the effect on the covariance matrix is a pure rotation

$$C \Rightarrow O^T C O$$

where $$O = \bigoplus_\alpha \begin{bmatrix} \cos\theta_\alpha & -\sin\theta_\alpha \\ \sin\theta_\alpha & \cos\theta_\alpha \end{bmatrix}$$

where θ_α are the eigenvalues of θ_{ij}

This gives the expected results of just causing a rotation in phase space without affecting the fluctuations in any quadrature.

- For squeezing

$$C \Rightarrow O_1 D O_2 C O_2^T D O_1^T$$

where $D = \bigoplus \begin{bmatrix} e^{2r_j} & 0 \\ 0 & e^{-2r_j} \end{bmatrix}$

Here O_2 rotates the quadratures to be squeezed along the X-P axis, D performs the squeezing, and O_1 rotates the final ellipse into place, as expected.

(g) Consider a single mode to start (this easily generalizes for multi-mode case)

Input covariance matrix for vacuum

$$C = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C \Rightarrow O_1 D^2 O_1^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} e^{+2r} & 0 \\ 0 & e^{-2r} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} e^{2r}\cos^2\theta + e^{-2r}\sin^2\theta & \sinh(2r)\sin 2\theta \\ \sinh(2r)\sin 2\theta & e^{-2r}\cos^2\theta + e^{2r}\sin^2\theta \end{bmatrix} \checkmark$$